Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE CONSTITUTION OF ALGEBRAIC ABILITIES[1]

By EDWARD L. THORNDIKE
INSTITUTE OF EDUCATIONAL RESEARCH, Teachers' College, Columbia University

The abilities acquired by a year's study of algebra may be constituted in very many different ways. A mathematician might think of the pupil as acquiring them by learning a few principles of notation, the laws of signs, the theory of exponents, the axioms, and the general rule that you can operate with literal numbers as with ordinary numbers. A mathematician who knew nothing of the conventional treatment of algebra in schools might begin to teach algebra as follows:

Let a, b and c represent any three numbers. Let $+$, $-$, $\times$, $\div$, the fraction line, $\sqrt{}$, $\sqrt[3]{}$, ^{2} and ^{3} be used as in arithmetic. Let ab or $a.b$ mean $a \times b$. Let ( ) mean that the expression within is to be treated as one number.

$+a = 0 + a \qquad +b = 0 + b \qquad +c = 0 + c$

$-a = 0 - a \qquad -b = 0 - b \qquad -c = 0 - c$

Adding $-a$ is the same as subtracting $+a$

Subtracting $-a$ is the same as adding $+a$

$(+a)(+b) = +ab$

$(+a)(-b) = -ab$

$(-a)(+b) = -ab$

$(-a)(-b) = +ab$

$\frac{+a}{+a} = +1$

$\frac{+a}{-a} = -1$

$\frac{-a}{+a} = -1$

$\frac{-a}{-a} = +1$

If equals are added to equals the results are equal.

If $a + b = c \qquad a + b + d = c + d$

If equals are subtracted from equals the results are equal.

If $a + b = c \qquad a + b - d = c - d$

and so on.

---

[1] The studies reported in this article were made possible by a grant from The Commonwealth Fund.

Such a straightforward general treatment has a good deal in its favor, in the way of brevity and dignity. The experience of teaching, however, shows that algebraic abilities are not constituted in the minds of the pupils, out of a few general sweeping laws. No text-book or teacher of today, for example, would dare to assume that pupils who had been taught as above would be sure to understand that $(c - d) - (e + f)$ means "Subtract $e + f$ from $c - d$," or that

$$\frac{cd}{a} - \sqrt{a + b + c}$$

means "Subtract $\sqrt{a + b + c}$ from $cd \div a$," or even that $7cd^2 - 4cd^2$ means "Subtract $4cd^2$ from $7cd^2$."

The customary approved teaching of today builds up these abilities out of many detailed abilities. The pupil learns the meanings of about two hundred and fifty terms, such as:

Abscissa
Absolute term
Absolute value
Aggregation
Algebraic addition
Algebraic expression
" number
" product
" solution
Algebraic subtraction

Antilogarithm
Applying a formula
Ascending powers
Axes
Axiom
Base (distinct from power)
Binomial
Binomial theorem
Brace

He learns about one hundred and fifty rules, such as:

Like roots of equals are equal.

To add a positive number to a negative number take the difference of their absolute values and prefix the sign of the numerically greater number.

$a - 0 = a$

$0 - a = -a$

To add similar monomials find the algebraic sum of the coefficients of the common factor and prefix this sum to the common factor.

He forms many habits, either as applications of these rules or as accessories acquired in the course of computation and problem solving. For example, the principle, " We can represent numbers by letters " develops into at least ten distinct habits of thought, namely:

1. A letter may mean a particular number of things, like men, boys, or eggs.
2. A letter may mean a particular number of units, like cents, quarts, feet.
3. A letter may mean any one of a number of numbers, like the number of dollars in the cost of any number of suits of clothes of a certain sort, or the number of square feet in any rectangle.
4. A letter may mean any number, as in $$(p + q)\ (p - q) = p^2 - q^2$$
5. If you call a certain number p, you may call 3 times that number q or r or s or any letter except p that you please, but it is commonly useful to call 3 times that number $3p$.
6. If you call a certain number p you may call 3 more than that number any letter except p that you please, but it is commonly useful to call it $p + 3$.
7. 7, 8 and 9. The same principles of consistency and utility with $p + 3$, $p - 3$, $\frac{p}{3}$ and $\frac{3}{p}$.
10. If we call a certain number (say, the profit Mr. A. made in Jan., 1922) p, we don't call it anything else and don't call p something different so long as we are thinking about that particular problem.

In spite of all our experience in teaching algebra we do not seem to have found the optimum constitution of algebraic abilities. Reformers like Rugg and Clark and Nunn not only eliminate certain abilities, change the emphasis on those which they retain, and add new ones; they also constitute the retained abilities in different ways. They would be the first to expect that other desirable changes will be found by further experimentation and improved insights.

To the psychologist who tries to follow through the mental operations of pupils from their first solutions of simple formulae to their comprehension of the parabolic relation, the binomial theorem, and the like, there seem to be many promising invitations to experiment, and even many cases where improvements can be made at once by a straightforward application of the laws of learning. In the remainder of this article we shall note some of the general features of the constitution of these abilities, namely the provision for habits now neglected, the elimination of unnecessary habits, and the use of adaptable, even of loose, habits in place of inflexible rules which in actual learning have to be broken.

PROVISION FOR HABITS NOW NEGLECTED

The hundred and fifty rules do not cover all that the pupil must know and do. For example, throughout algebra the pupil has to decide when to indicate an operation only and when to carry it through according to some known routine. Suppose that he is faced with the need of dividing a by b, a^2 by a, a by a^2, and $\sqrt{625}$ by $\sqrt{10}$. In the first case he must only indicate $\frac{a}{b}$; in the second he must subtract with the exponents and write a; in the third case he must (according to usual methods of teaching) not subtract with the exponents, but indicate the division as $\frac{a}{a^2}$ and then divide each term by a; in the last case he is again customarily taught to indicate the division as $\frac{\sqrt{625}}{\sqrt{10}}$ and then proceed to $\sqrt{\frac{625}{10}}$ or $\sqrt{62.5}$, or to $\frac{25}{\sqrt{10}}$ or still worse to $\frac{\sqrt{625} \times \sqrt{10}}{10}$ or $\frac{\sqrt{6250}}{10}$. It would seem worth while to provide definitely for each useful habit, where there is some one procedure that is best, and to provide also a clear understanding that in certain cases you can do one of a number of things, any one of which is right so far as it goes, but some one of which will be best according to circumstances.

To take another illustration, in teaching division of a monomial by a monomial it is customary to use almost or quite exclusively tasks like $\frac{a^2}{a}$, $\frac{ab^2c^3}{ab^2c}$ and $\frac{27a^3b^3c}{9\,a^2b}$ where the quotient is integral—where no factor is left below the fraction line. $\frac{a}{a^2}$, $\frac{ab^2c}{ab^2c^3}$ and $\frac{9a^2b}{27a^3b^3c}$ commonly do not occur at all in the systematic treatment of division, and occur very rarely in the reduction of fractions to simplest form. In a census of all the work, oral and written, computation and problem solving, provided for the first year's study of algebra in three standard instruments of instruction, we find the following enormous disparity between samples of the two types of task:

Let a, b, c, etc., represent numerals.

Let x, y, z, etc., represent literal numbers.

Then the averages for the three text-books are:

$$\frac{ax}{b} \text{ and } \frac{axy}{b} \text{ occur 848 times.}$$

$$\frac{a}{bx} \text{ and } \frac{a}{bxy} \text{ occur 2 times.}$$

$$\frac{ax}{x}, \frac{x^2}{x}, \text{ and } \frac{x^3}{x} \text{ occur 245 times.}$$

$$\frac{x}{ax}, \frac{x}{x^2}, \text{ and } \frac{x}{x^3} \text{ occur 8 times.}$$

This is an inadequate preparation for the actual use of algebra in later mathematical study, science, or the general work of life. The larger numeral, the persisting literal factor, and the higher power probably do go in the numerator oftener than in the denominator, but not in the ratio of 110 to 1. If fifteen-year-old boys and girls have a certain result happen 110 times as often as its opposite, they tend to think of that opposite as impossible or wrong. They will feel no surety when any division comes out with a balance in the denominator.

The customary procedure is due to the tradition that considered fractions as a difficult matter, not to be touched in any way until all operations with integers had been mastered, and to the effort to fit the pupils' operations to the two rules of dividing by subtracting exponents and of reducing fractions by dividing both numerator and denominator by the same factor. The tradition is simply nonsense, a fallacious notion due to analogy with arithmetic and the idea that because fractions as a whole are hard, everything about them is hard. The fraction form is wisely used in the division of a monomial by a monomial. The plan of fitting the pupils' operations to the two rules has much in its favor and can reasonably be retained, the defect in the pupils' habits being remedied by extending the notion of division when reduction of fractions is learned, and by giving suitable practice then and thereafter. It would, however, perhaps be still more effective to make the procedure general from the beginning by introducing —1, —2, —3 and 0 as exponents, teaching their meanings, and permitting either a^{-1} or $\frac{1}{a}$ as an answer to $\frac{a}{a^2}$ or $\frac{a^2}{a^3}$, and similarly for all factors remaining below the fraction line. Teachers in general will be shocked at the proposal to teach anything about negative exponents at this stage, but the most acute teachers may not be. For some of them have perceived by intuition and experience, what psychologists infer from general principles, that it is often very advantageous to get used to a few elements of a topic or doctrine long before the doctrine as a whole is systematically taught. Thus in arithmetic we now teach certain facts about the addition of fractions two or three years before the general treatment of the addition of fractions, and teach addition, subtraction, multiplication and division with United States money long before the general treatment of decimals, and do so with great profit, both to the early work and to the general treatment when it comes.

THE ELIMINATION OF UNNECESSARY HABITS

We teachers and learned people are obsessed by the tendency to learn everything about anything and teach everything about anything that attracts our intellects. The former tendency is, in spite of certain pedantries, perhaps the greatest blessing the world has. The latter is also a blessing on the whole, but it can be made a greater blessing if we control it. It has needed control in algebra. The educational reformers have had to work hard to convince teachers of mathematics that it is not wise or humane or scientific to burden the learning of algebra by children with all the factorizations ingenuity can devise, or to hide $x^2 — y^2$ under all the disguises through which mathematical acuity can penetrate.

Consider the case of the teaching of radicals as a sample. Suppose that we eliminated all the customary work with radicals and the customary general systematic treatment of powers during the first year, and instead taught the facts now taught much later concerning fractional exponents and negative exponents, plus the following:

1. $\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$; if $a = b$, $a^m = b^m$,—the rule for principal roots.

2. $\sqrt{a}$ means $a^{\frac{1}{2}}$ $\sqrt[3]{a}$ means $a^{\frac{1}{3}}$ $\sqrt[4]{a}$ means $a^{\frac{1}{4}}$, etc.
$\sqrt{a^3}$ means $a^{\frac{3}{2}}$ $\sqrt{a^4}$ means $a^{\frac{4}{2}}$ $\sqrt{a^5}$ means $a^{\frac{5}{2}}$, etc.
$\sqrt[3]{a^2}$ means $a^{\frac{2}{3}}$ $\sqrt[3]{a^3}$ means $a^{\frac{3}{3}}$ $\sqrt[3]{a^4}$ means $a^{\frac{4}{3}}$, etc.

3. If in any task you meet an expression with $\sqrt{\ }$ or $\sqrt[2]{\ }$ or $\sqrt[3]{\ }$, change it to an equivalent expression with the proper exponents (and with parentheses if necessary).

4. If, after multiplying, dividing, raising to powers and finding roots by proper use of the exponents, any arithmetical number still has an exponent other than 1, evaluate by using tables of powers and roots, or by logarithms, or by trial and correction if you cannot get the necessary tables.

5. If, however, you see some way of shortening the work, as by $(5^{\frac{1}{2}} + 7^{\frac{1}{2}})(5^{\frac{1}{2}} - 7^{\frac{1}{2}}) = 5 - 7$, take it.

This suggestion will, like the one on page 410 shock many teachers. They will think, or rather feel, that the general procedure with exponents is too hard to teach thus early, that $\sqrt{\ }$ should precede, not follow $^{\frac{1}{2}}$, and that it will seriously mutilate algebra to omit such stimuli to ingenuity as $\sqrt[3]{9}(27)\sqrt{3}$, or $(\sqrt{8} + \sqrt{7})(8 + 2\sqrt{7})$, or $\sqrt[4]{\frac{2}{3}}$, or $\frac{2 - \sqrt{5}}{3 + \sqrt{2}}$ or to replace them by equivalents with fractional indices whose simplifications are aided by tables.

These objections are instructive. The difficulty pupils have in learning the general procedure with fractional and negative exponents is largely of our own manufacture. In the ordinary course of events what they learn about roots, radical signs, surds, and rationalizing hinders them in learning the general treatment. "If it is 2 in $\sqrt{\ }$ how can it be ½? If it requires two distinct operations in $\sqrt[3]{8^2}$ how can you express it as a single fraction $8^{\frac{2}{3}}$? And why does it turn bottom side up? I was taught to turn $\sqrt[3]{a^4}$ into two a's $(a\sqrt[3]{a})$ as soon as I saw it. Is it reasonable for me to call it $a^{\frac{4}{3}}$ now? In any case why not leave me to do it in peace in the old way that I learned at such cost?'' Such is the unconscious argument of the pupil's nervous system. In the ordinary type of 'explanation' of the procedure with exponents, we are prone to add to its difficulties for all save the gifted pupils. We explain how it can be true, justify it, and present it in the highly abstract rules

$a^m \times a^n = a^{m+n}$ whatever m and n may be
$a^m \div a^n = a^{m-n}$ whatever m and n may be
$(a^m)^n = a^{mn}$ whatever m and n may be

and quiz the pupils lest they be not convinced.

Now the difficulty is not that the pupils feel logical objections, or any intellectual shock at the innovation. Unfortunately many of them would not rebel intellectually if they were told that hereafter a^2 would always mean $10a$, a^3 would mean $100a$, and so on. Nor are they helped greatly to understand the reasonableness of the new system by $a^m \times a^n = a^{m+n}$, etc. Their chief difficulty is that they are not *used to it* and do not realize *what* they are doing. They feel lost and dizzy with these strange exponents. They need first just to become at home with $(a^{\frac{1}{2}} + b^{\frac{1}{2}})$ $(a^{\frac{1}{2}} - b^{\frac{1}{2}})$ and the like. They need, second, to see how the results do turn out by this new procedure, to see that $a^{\frac{5}{2}}$ means $a^{2\frac{1}{2}}$ and is about half way between a^2 and a^3, that 10^{-4} is $\frac{1}{10000}$, and the like. Such series as the following are useful for this:

$10^4 = 10000$
$10^3 = 1000$
$10^{2\frac{1}{2}} = 316 +$
$10^2 = 100$
$10^{1\frac{3}{4}} = 56.2$
$10^{1\frac{1}{2}} = 31.6$
$10^{1\frac{1}{4}} = 17.8$
$10^1 = 10$
$10^{\frac{3}{4}} = 5.62$
$10^{\frac{2}{3}} = 4.64$

$10^{\frac{1}{2}} = 3.16$
$10^{\frac{1}{3}} = 2.15$
$10^{\frac{1}{4}} = 1.78$
$4^3 = 64$
$4^5 = 32$
$4^{\frac{4}{2}} = 16$
$4^{\frac{3}{2}} = 8$
$4^{\frac{2}{2}} = 4$
$4^{\frac{1}{2}} = 2$
$9^3 = 729$

$9^{\frac{8}{3}} = 350 +$
$9^{\frac{7}{3}} = 168 +$
$9^{\frac{6}{3}} = 81$
$9^{\frac{5}{3}} = 38.9$
$9^{\frac{4}{3}} = 18.7$
$9^{\frac{3}{3}} = 9$
$9^{\frac{2}{3}} = 4.32$
$9^{\frac{1}{3}} = 2.08$

They need, in the third place, to have enough practice in the operations so that they can do them correctly and have their minds free to consider what they are doing, and what comes of it, and why it is a useful thing to do, and how the formulas, $a^m \times a^n = a^{m+n}$ and the rest, sum up in a beautiful set of rules a great many operations which they have learned to perform and to trust as sure means of obtaining correct results. The very pupils to whom these formulas are baffling verbal edicts at the beginning of the learning may, after enough instructive practice and checking, find them admirable summaries of what they have learned.

The second objection, to the effect that $\sqrt{}$ and $\sqrt[3]{}$ and $\sqrt[4]{}$ should precede ${}^{\frac{1}{2}}$ and ${}^{\frac{1}{3}}$ and ${}^{\frac{1}{4}}$, can hardly have any other origin than mistaking what is familiar to us as what is easy and natural[1]. ${}^{\frac{1}{2}}\ {}^{\frac{1}{3}}\ {}^{\frac{1}{4}}$ are better symbols than $\sqrt{}\ \sqrt[3]{}\ \sqrt[4]{}$ in every way whatsoever save two, for all pupils whatsoever. Their two demerits are, of course, the likelihood of confusing them with common fractions, and the more common use of $\sqrt{}\ \sqrt[3]{}\ \sqrt[4]{}$.

Pupils in high school are protected against the confusion with fractions by long habituation to a^2, a^3, a^4, etc. They should learn to understand both types of symbol, but ${}^{\frac{1}{2}}$, ${}^{\frac{1}{3}}$, etc., should precede. These should indeed be used in the first treatment of roots in algebra, whenever that may be, the pupil being taught that the square root of a in algebra is written $a^{\frac{1}{2}}$, or, sometimes, $\sqrt{a}$.

The third objection was that the rule permitting students to get rid of certain surds by direct use of tables, logarithms or computation instead of searching for clever ways to rationalize would rob algebra of a certain portion of its intellectuality. It would, but the general addition of intellectuality by making the treatment of radicals as a whole a logical and straightforward application of the theory of exponents, far outweighs the loss. Radicals as a whole as now taught are a confused set of maxims plus a bag of clever tricks.

---

[1] It seems probable that the antipathy to fractional exponents, founded on their unfamiliarity, is made much worse by the strain that commonly attends reading them because the type used is so small. The traditions of the printer in this respect are very bad, and the authors of text books have weakly given way to him. Fractional exponents should, of course, be printed in type that is easily legible.

On the whole it seems conservative to estimate that at least three fourths of the time now given to radicals could be saved with no loss, but a net gain to the pupil's development.[1]

ADAPTABLE HABITS VERSUS INFLEXIBLE RULES

The rigor of its definitions and the universality of its rules are among algebra's chief merits, and we should cherish them. We do not, however, dignify algebra by claiming a universality when it is really lacking, nor by asserting something rigorously and evading it soon thereafter. When the definition works only part of the time and the rule is helpful for only some of the cases, it may be better to organize the learning as a set of rather loose and opportunistic habits than to insist on the rule and then qualify and amend and even break it.

Consider these two definitions and these two rules:

(1) If an expression is separated into two factors, either factor is called the coefficient of the other.

(2) Terms which differ in no respect or only in their coefficients are called like terms.

(3) To add like terms add the coefficients of the common factor and prefix this sum to the common factor.

(4) To add polynomials we write like terms in the same column and add these terms, writing their sums as a polynomial.

Consider $4ab^2c$. By the definition (1), b^2 is called the coefficient of $4ac$, and it would be reasonable to call it so, but in actual fact it isn't so called. $4ab^2$ by the definition is the coefficient of c.

Consider $5a^2b^3c$, regarding one factor as $5a^2b^3$ and the other as c. By the definition (2), $4ab^2c$ and $5a^2b^3c$ are called like terms, and it would be reasonable to call them so, but teachers would not call them so in one case out of fifty, and might rebuke pupils if they so called them.

---

[1] It may be noted that an appreciation of the desirability of using the general procedure with exponents in the computations with radicals is shown by certain teachers and authors of textbooks who insert more or less of the general treatment along with the older teachings as an *alternative* way of handling the computations. This, however, seems to be of little or no avail in lightening the pupil's load. We do not wish to have him learn the inferior way and that there is a better way, but to be blissfully ignorant of the inferior way. We wish him only to turn $\sqrt{}$ into exponential form and to know the common equivalents so that he can read books using $\sqrt{}$. Apart from that, the less he does with $\sqrt[3]{a^2}$, etc., and the more exclusively he works with $a^{\frac{1}{3}}$ $a^{\frac{2}{3}}$, $a^{\frac{3}{2}}$, etc., the better pleased we should be.

Consider:—

Add $3abc + 4ab^2c + 5a^2b^2c$ to $6abc + \frac{16abc^2}{c} + 7cef + 8abc$.

Applying rules (3) and (4), which shall we put in the same column, all of the terms or only the first and last, or the first, last and second from the last?

In fact it is not the definitions and rules, but the habits developed by experience with particular numbers and expressions that teach a pupil what to regard as the coefficients, and what terms to collect together in addition. A practical joker could arrange a set of exercises in early addition that would force a conscientious pupil who tried to follow the definitions and rules into utter bewilderment. Would it not be better to build up useful habits of addition and subtraction without the pretence that they are logical deductions from the definitions and rules?

" An algebraic expression in which the parts are not separated by + or — is called a monomial " is given as a rigorous definition; but later the pupil has to consider $3(ab - cd)$ as a monomial; and later he must treat $\frac{12a^2b^2c}{3\,abc}$ as two monomials though there is no separation by + or —.

A special case of importance is the erection of technically convenient procedures like arranging in ascending or descending powers before multiplying or dividing, writing like terms under one another, or beginning a monomial with its numerical factor, into fixed rules on a level with the most general axioms and principles. Every such rule which a pupil finds that he can break (as he can all such) without getting wrong answers means a risk that he will lose respect for and confidence in the really imperative rules. If we leave to habit everything that can be done as well by habit, we gain an added dignity for the matters that really are matters of principle. For example, it seems sound psychology to teach " If equals are added to equals the results are equal" as a rule, but one should teach "transposing," universally valid though it is, as a convenient habit. It is not because he does not value rules and principles in algebra that the psychologist often prefers to use habits instead; it is rather because he does value principles and does not wish them to be misused and cheapened.